MATH2050C Assignment 4

Deadline: Feb 4, 2015 .

Hand in: 3.2 no. 14b, 16d, 19d; Suppl Problems no 3, 4.

Section 3.2 no. 6cd, 10a, 11a, 14ab, 16cd, 19ad, 20, 21.

Supplementary Problems

1. Suppose that $x_n \to x, x_n \geq 0$. Show that $x_n^{p/q} \to x^{p/q}$ for $p, q \in \mathbb{N}$.
2. Find (a) $\lim_{n\to\infty} n^{1/n^2}$ and (b) $\lim_{n\to\infty} (\frac{n+5}{n-4})^{(n+1)/n}, n \geq 5$.
3. Determine the limit of
$$\left(1 - \frac{a}{n^2}\right)^n \ , \ a > 0 \ .$$
Hint: Use Bernoulli's inequality and the Squeeze Theorem.
4. Suppose that $\lim_{n\to\infty} x_n = x$. Prove that
$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = x \ .$$

See next page for more.

More on Limits

Recall last time we proved

Proposition 3.1 Whenever there is some $\{c_n\}, c_n \to 0$ as $n \to \infty$ satisfying $|x_n - x| \le c_n$, $\lim_{n\to\infty} x_n = x$.

As the consequences of this proposition, we have

Proposition 4.1(Ratio Test) Let $\{x_n\}$ be a sequence of positive numbers. Suppose that there exists $\gamma \in [0,1)$ such that $x_{n+1}/x_n \le \gamma$ for all $n \ge n_0$. Then $\lim_{n\to\infty} x_n = 0$.

Proof. WLOG take $n_0 = 1$. We have

$$x_n = \frac{x_n}{x_{n-1}} \frac{x_{n-1}}{x_{n-2}} \cdots \frac{x_2}{x_1} x_1 \le x_1 \gamma^{n-1} \ .$$

From $\lim_{n\to\infty} \gamma^n = 0$ and Proposition 3.1 we draw the desired conclusion.

Remark. In our textbook Theorem 3.2.11 the assumption $\lim_{n\to\infty} x_{n+1}/x_n = L \in [0,1)$ is used instead of $x_{n+1}/x_n \le \gamma$. In fact, when $\lim_{n\to\infty} x_{n+1}/x_n = L \in [0,1)$ holds, we can choose a small $\varepsilon_0 > 0$ so that $\gamma = L + \varepsilon_0 L < 1$ satisfies $x_{n+1}/x_n \le \gamma$, $\forall n \ge n_0$ for some n_0.

Example 4.1 Show that $\lim_{n\to\infty} a^n/n! = 0$ for any $a > 0$. We have $x_{n+1}/x_n = a/(n+1) \to 0$ as $n \to \infty$. By the ratio test, $\lim_{n\to\infty} a^n/n! = 0$.

Proposition 4.2 (Root Test) Let $\{x_n\}$ be a positive sequence. Suppose that there exists $\gamma \in [0,1)$ such that $x_n^{1/n} \le \gamma$ for all $n \ge n_0$. Then $\lim_{n\to\infty} x_n = 0$.

The proof is left as exercise.

Next we discuss how the process of taking limits interacts with the algebraic operations of the real number system. The main result is contained in the limit theorems, see Theorem 3.2.3.

Example 4.2 Find $\lim_{n\to\infty} \dfrac{3n^2 + 5n - 12}{5n^2 - 2n + 13}$. A quick proof is furnished by using the limit theorem. We have

$$\begin{aligned} \lim_{n\to\infty} \frac{3n^2 + 5n - 12}{5n^2 - 2n + 13} &= \lim_{n\to\infty} \frac{3n^2}{5n^2} \frac{1 + 5/3n - 4/n^2}{1 - 2/5n + 13/5n^2} \\ &= \frac{3}{5} \frac{\lim_{n\to\infty}(1 + 5/3n - 4/n^2)}{\lim_{n\to\infty}(1 - 2/5n + 13/5n^2)} \\ &= \frac{3}{5} \ . \end{aligned}$$

The process of taking limits also interacts well with the ordering. See the Squeeze Theorem 3.2. Note that Proposition 3.1 is a special case of this theorem.